with a similar equation for (E), but there is no simple geometrical method of attacking this wilderness of ordinates. The y's stand at every point in the x-plane, but they have various slants towards the fourth direction of space; all we notice is, that those of the same slant or tilt have their feet on an hyperbola around O as center. Methods of function-theory must here be used.

Conclusion.—One primary advantage of treating hyperbolic functions from this general standpoint lies, in the writer's judgment, in the relegation of i to its proper subordinate position; protocyclic functions are real, and represent geometric entities from the very start; the generalization that occurs in their definition is seen to be geometrically appropriate. The theoretic interest is mainly in the fact that circle-trigonometry and hyperbola-trigonometry can now be comprehended in the wider topic of the trigonometry of the central conics.

A PROOF OF A THEOREM OF COMPOUND PROBABILITIES.

By UGO BROGGI,[1] Universidad De La Plata.

1. The definition of a geometric probability is well known. We call the probability, $\phi(a, b)$, that a real number, x, belongs to the interval (a, b) a real function, ϕ, of a and b which satisfies the conditions

$$0 \leq \phi(a, b) \leq 1,$$

and, if $a \leq c \leq b$,

$$\phi(a, b) = \phi(a, c) + \phi(c, b).$$

It would be an easy matter, but beside our purpose, to generalize the definition, and apply it not only to intervals but also to more general one- or more-dimensional sets of points.

Since the definition of our new probability is based upon the theorem of total probabilities, it is evident that it must agree with that theorem.

We cannot without proof say the same of the theorem of compound probabilities, which asserts that if p is the probability that x belongs to (a, b), and π the probability that y belongs to (α, β) and there is no relation between x and y, the probability P that the two-dimensional number (x, y) belongs to the region

$$a \leq x \leq b, \qquad \alpha \leq y \leq \beta,$$

is expressed by $p\pi$.

It is a surprising fact that nowhere do we find the demonstration of this theorem, one which is evidently fundamental and without which the theory of errors and the kinetic theory of gases would come to nothing.

We propose to give here the missing demonstration.

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2. Suppose a and α to be constant, and put

$$b = a + h, \quad \beta = \alpha + k; \quad p = p(a, h), \quad \pi = \pi(\alpha, k); \quad P = P(a, \alpha; h, k).$$

It is evident that

$$p \geqq P, \qquad \pi \geqq P,$$

and, when $h_2 > h_1 > 0$, $k_2 > k_1 > 0$,

$$p(a, h_2) - p(a, h_1) \geqq P(a, \alpha; h_2, k) - P(a, \alpha; h_1, k) \geqq 0$$

$$\pi(\alpha, k_2) - \pi(\alpha, k_1) \geqq P(a, \alpha; h, k_2) - P(a, \alpha; h, k_1) \geqq 0.$$

P cannot be an increasing function of h or of k unless p and π are increasing functions of their arguments. If p and π are increasing functions of h and of k, there is only one value of h or of k corresponding to a given value of p or of π, respectively. Hence in

$$P = P(a, \alpha; h, k)$$

we can express h and k through p and π and have

$$P = F(p, \pi).$$

If

$$\pi_1 = \pi(\alpha, k_1), \qquad \pi_2 = \pi - \pi_1 \qquad (k_1 \leqq k)$$

we have also, from the theorem of total probabilities

$$P = F(p, \pi_1) + F(p, \pi_2)$$

and therefore[1]

$$P = p\pi.$$

3. Let us now suppose that the inequality

$$p(a, h_1) < p(a, h_2) \qquad (h_1 < h_2)$$

holds true only for $h \leqq h_1$, and that two values h_1 and h_2 exist, such that

$$p(a, h_1) = p(a, h_2).$$

Then for every h belonging to the interval (h_1, h_2) we must have

$$p(a, h) = p(a, h_1).$$

$$0 = p(a, h) - p(a, h_1) \geqq P(a, \alpha; h, k) - P(a, \alpha; h_1, k) \geqq 0$$

$$P(a, \alpha; h, k) = P(a, \alpha; h_1, k) = p(a, h)\pi(\alpha, k).$$

The theorem must still hold true when we have identically

$$p(a, h) = 0$$

[1] Broggi, "Il teorema della probabilità composta," in *Rendiconti del Circolo Matematico di Palermo*, Vol. 28 (1909), 245–247.

for $h \leqq h_1$. In this case

$$p \gtreqq P \geqq 0, \qquad P = p\pi = 0.$$

4. We can summarize the results here obtained by saying that the theorem of compound probabilities holds true when p and π are increasing functions of their arguments, or increase in certain intervals and are constant elsewhere. But this condition is always realized, as we cannot have

$$p(a, h_1) = p(a, h_2), \qquad \pi(\alpha, k_1) = \pi(\alpha, k_2)$$

unless the functions p and π are constant in the intervals (h_1, h_2) and (k_1, k_2) respectively.

BITS OF HISTORY ABOUT TWO COMMON MATHEMATICAL TERMS.

By G. A. MILLER, University of Illinois.

In 1841 A. L. Cauchy defined the term *indicator* (indicateur) corresponding to a given modulus n as the exponent to which a positive integer m relatively prime to n belongs mod n, and gave tables for the determination of the maximum indicator I corresponding to a given modulus n, where n may be replaced successively by a series of positive integers. For instance,

$n = 2$	3	4	5	6	7	8	9	10	11	12	13
$I = 1$	2	2	4	2	6	2	6	4	10	2	12

About four years later E. Prouhet published a note in volume 5 of the *Nouvelles Annales de Mathématiques*, page 75, in which he defined the term indicator of n as the number $\phi(n)$ of the positive integers less than n and prime to n. He added incorrectly that this term had not been employed previously in mathematics. It seems somewhat singular that this later definition came into common use, especially since Cauchy was a more noted mathematician than Prouhet and this definition was due to ignorance on the part of its author.

In the language of group theory Cauchy's definition of indicator is equivalent to that of the order of the totitives of n, while that of Prouhet is equivalent to that of the order of the group formed by these totitives mod n. The I of n is therefore always a divisor of $\varphi(n)$ and a necessary and sufficient condition that I of n is equal to $\phi(n)$ is that n has a primitive root, or that the group of totitives of n is cyclic. In general, the value of I is equivalent to the order of the largest cyclic subgroup contained in this group.

As an instance where a later definition of a common term had an entirely different fate we may refer to the definition of *simple group* found on page 65, volume 20, *Proceedings of the London Mathematical Society*. According to this definition a group is called *simple* when it is both cyclic and its order is a power of a prime number, while the common definition of simple group as a group which